# REPORT 1179

## A NOTE ON SECONDARY FLOW IN ROTATING RADIAL CHANNELS[1]

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### SUMMARY

*A general vector differential equation for the vorticity component parallel to a streamline is derived for steady, nonviscous, and incompressible flow in a rotating system. This equation is then simplified by restricting it to rotating radial channels and by making further simplifying assumptions. The simplified equation is used to solve for the secondary vorticity, the vorticity component parallel to the streamline, in three special cases involving different streamtube geometries; the results are presented in a series of figures. The secondary vorticity is shown to decrease with decreased absolute angular velocity of the fluid, decreased inlet total-pressure gradient, decreased length of relative flow path, and increased relative velocity.*


### INTRODUCTION

When a fluid with a total-pressure gradient normal to the osculating plane of the streamline follows a curved path, velocities are induced in the surfaces normal to the irrotational or "through" flow direction. These induced velocities or secondary flows are recognized as the source of several types of loss in turbomachinery. For example, losses may occur because of (1) boundary-layer separation caused by the transfer of low-energy air to regions of decelerating flow, (2) improper angles of attack in the blade rows of compressors and turbines, and (3) viscous dissipation and unrecoverability of the kinetic energy involved in the secondary flow.

Because secondary flows are an important source of loss, considerable analytic work has been done on this phenomenon in stationary curved channels by Squire and Winter (ref. 1), Hawthorne (ref. 2), Kronauer (ref. 3), and others. The general method used in these analyses is to compute the vorticity component parallel to the actual streamline, the secondary vorticity, caused by a variation in inlet total pressure across the passage normal to the osculating plane of the streamline. This secondary vorticity is computed because it is an indication of the magnitude of the secondary flow. In addition, Kronauer (ref. 3), using a similar method, has computed this vorticity component in an axial-flow rotating channel such as that in an axial-flow compressor. However, no work has been done on radial-flow rotating channels, such as in a centrifugal-type impeller.

This report extends the work of Hawthorne and the others to a rotating radial channel. The results of the analysis are indications of the qualitative trends of variables which affect the secondary vorticity and therefore the secondary flow. The purpose thus is not to obtain exact data but rather to obtain some insight into the problem of secondary flows in rotating radial channels. It is hoped that determining the relative importance of some of the variables involved will lead to a better understanding of the flow in centrifugal-type impellers and similar flow machinery. Calculations are carried out to show the effects of these variables in several streamtubes with various geometries.

### METHOD OF ANALYSIS

In this analysis the distribution along a streamline of the secondary vorticity (component of the vorticity parallel to the streamline) is computed for rotating radial channels.

#### PRELIMINARY CONSIDERATIONS

For steady three-dimensional flow of a nonviscous fluid with nonuniform total-pressure distribution through a stationary channel, the streamlines and vortex lines lie on surfaces of constant total pressure (ref. 4, p. 244) called Bernoulli surfaces. The total-pressure gradient $\nabla P$ is necessarily normal to the Bernoulli surfaces, and in general these surfaces are curved in space. The vorticity vector $\bar{\zeta}$, which is tangent to the vortex line, is defined by

$$\bar{\zeta}=\nabla\times\bar{V}$$

where $\bar{V}$ is the absolute velocity vector tangent to the streamline. From the equation of motion the vectors $\bar{V}$, $\bar{\zeta}$, and $\nabla P$ are related by

$$\bar{V}\times\bar{\zeta}=\nabla P$$

Since $\nabla P$ is normal to the Bernoulli surface, this equation shows that the streamlines and vortex lines lie on Bernoulli surfaces. In general, the vorticity vector $\bar{\zeta}$ has components tangent and normal to the absolute velocity vector $\bar{V}$, $\zeta\cos\psi$ and $\zeta\sin\psi$, respectively, where $\psi$ is the angle between the vortex line and the streamline (see fig. 1). The vorticity component tangent to $\bar{V}$ is the secondary vorticity, and it is this component of vorticity that is an indication of the magnitude of secondary flow. Between any two points on the Bernoulli surface there exists a geodesic or path of minimum distance. In general, the streamline is not a path

[1] Supersedes NACA TN 3013, "A Note on Secondary Flow in Rotating Radial Channels," by James J. Kramer and John D. Stanitz, 1953.

368555—56——35

of minimum distance on the Bernoulli surface, that is, the streamline is not a geodesic; therefore the principal normal $\overline{N}$ of the streamline is inclined at an angle $\varphi$ (fig. 1) from the normal to the Bernoulli surface $\nabla P$.

If the streamline is not a geodesic on the Bernoulli surface, the secondary vorticity increases in the direction of the flow (ref. 2). This condition exists if $\varphi$ is not zero, that is, if $\nabla P$ has a component normal to the osculating plane of the streamline, the plane containing the velocity vector $\overline{V}$ and the principal normal $\overline{N}$. If the streamline is a geodesic, $\nabla P$ lies in the osculating plane and the secondary vorticity does not change along the streamline.

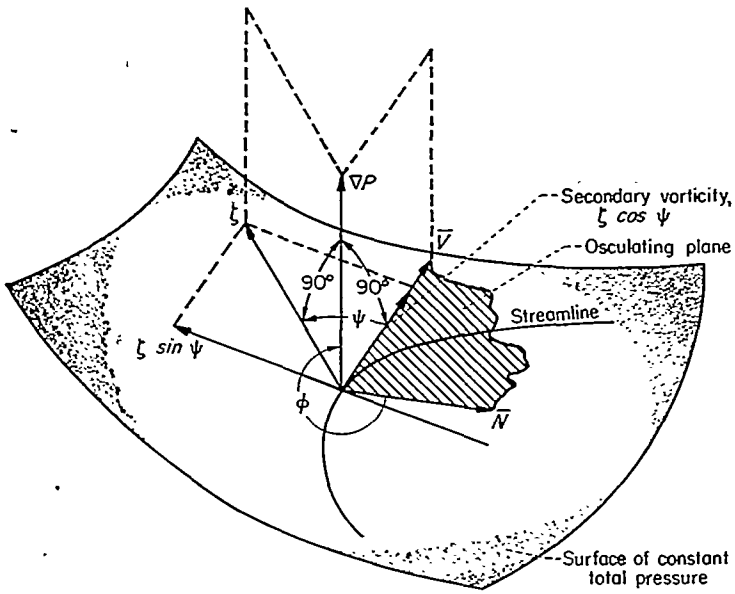



FIGURE 1.—Vector diagram for stationary channel.

The general assumptions and method used in the development of the equations for secondary vorticity in a rotating radial channel are outlined in the following sections.

**General assumptions.**—The flow is assumed to be steady, incompressible, and nonviscous. Although the fluid is assumed to be nonviscous in the region of the solution, a gradient in inlet total pressure normal to the flow planes is assumed that would have been caused by the generation of a thick boundary layer by viscous forces upstream of the region of solution, that is, upstream of the rotating channel. If it is assumed, as will be done herein, that the boundary-layer thickness is large compared with the passage height (i. e., viscous shear forces are low), then the fluid can be assumed nonviscous and the vorticity calculated from the equation of motion for an ideal fluid. This assumption is also made in references 1 to 3.

The flow entering the passage is assumed to be parallel and uniform except for the thick boundary layer which results in a total-pressure gradient normal to the osculating plane of the relative streamline. In a centrifugal-type impeller, this would be the boundary layer on the inner and outer shrouds. Thus, initially, the vorticity component parallel to the streamline is zero. A component of vorticity parallel to the streamline is generated when the streamline turns from its initial direction or is rotated in a rotating system.

**Outline of method.**—In order to evaluate the secondary vorticity along a streamline, the vorticity component parallel to a streamline is expressed as a function of the relative total vorticity and the relative velocity. This expression is simplified using the fact that the divergence of the total vorticity is zero. The total relative vorticity is eliminated from the equation by means of the equation of motion. The equation of continuity is used to express the gradient along a streamline of the magnitude of the vorticity component parallel to a streamline divided by the relative velocity as a function of the relative velocity, the angular velocity of the rotating system, and the gradient of the Bernoulli constant in the rotating system. This is the general vector differential equation for secondary vorticity in a rotating system. This general vector differential equation is then simplified by restricting it to rotating radial channels and by making further simplifying assumptions. In order to solve for the secondary vorticity by means of this equation, it is necessary to know the streamline pattern. As the actual streamline configuration is, of course, unknown at the outset of the problem, it is necessary to assume a realistic streamline pattern which from general considerations would probably not differ greatly from the actual pattern in order to obtain meaningful results. This assumed streamline configuration is based on a potential flow solution or some other approximation, and a first approximation to the secondary vorticity is obtained by means of the simplified differential equation for secondary vorticity. The secondary vorticity is solved for in several rotating radial streamtubes with various geometries using the simplified differential equation.

## GENERAL VECTOR DIFFERENTIAL EQUATION FOR SECONDARY VORTICITY

The various steps outlined in the development of the general vector differential equation for secondary vorticity are now presented in detail.

**Vorticity vector.**—The total relative vorticity is first expressed as the sum of its components parallel and normal to the streamline. Let $\overline{W}$ be the velocity vector relative to the rotating system and $\overline{\Omega}$, the relative vorticity vector equal to $\nabla\times\overline{W}$. (All symbols are defined in appendix A and are expressed in terms of dimensionless ratios.) The dimensionless relative velocity $\overline{W}$ is a ratio of the dimensional relative velocity to some characteristic speed of the channel $\omega r_t$. The angular velocity of the system is $\omega$ and $r_t$ is a reference distance from the axis of rotation to some point in the channel. The vorticity $\overline{\Omega}$ and the magnitude of its component $\xi$ parallel to a streamline are dimensionless, having been divided by $\omega$.

It can be seen from the expansion of the triple vector product $(\overline{W}\times\overline{\Omega})\times\overline{W}$ that

$$\overline{\Omega}=\frac{\overline{\Omega}\cdot\overline{W}}{\overline{W}\cdot\overline{W}}\overline{W}+\frac{(\overline{W}\times\overline{\Omega})\times\overline{W}}{\overline{W}\cdot\overline{W}}$$

From the definitions of the dot and cross products it follows that the terms on the right are equal to the components of vorticity parallel and normal, respectively, to a streamline. Since $\xi$ is the magnitude of the component of the vorticity parallel to a streamline, the preceding equation can be written

$$\bar{\Omega}=\frac{\xi}{\overline{W}}\overline{W}+\frac{(\overline{W}\times\bar{\Omega})\times\overline{W}}{\overline{W}\cdot\overline{W}} \tag{1}$$

where the absence of the bar indicates a scalar quantity. Because the divergence of the curl of any vector is zero, $\bar{\Omega}$ can be eliminated from the left side of equation (1) as follows:

$$\nabla\cdot\bar{\Omega}=\nabla\cdot(\nabla\times\overline{W})=0$$

so that equation (1) becomes

$$\nabla\cdot\frac{\xi}{\overline{W}}\overline{W}=\nabla\cdot\frac{\overline{W}\times(\overline{W}\times\bar{\Omega})}{\overline{W}\cdot\overline{W}} \tag{2}$$

In order to simplify equation (2) the equations of motion and continuity are introduced.

**Equation of motion.**—The equation of motion can be written (ref. 5, eq. (15))

$$\overline{W}\times\bar{\Omega}=\nabla H+\frac{2\bar{\omega}\times\overline{W}}{\omega} \tag{3}$$

where $H$ is the Bernoulli constant relative to a rotating system, defined symbolically as

$$H=\frac{p/\rho}{(\omega r_t)^2}+\frac{W^2}{2}-\frac{R^2}{2} \tag{4}$$

In equation (4), $p/\rho$ is the ratio of the static pressure to the fluid mass density, made dimensionless by dividing by $(\omega r_t)^2$, and $R$ is the dimensionless distance in the radial direction expressed as a ratio of $r_t$, as are all linear dimensions. The quantity $H$ is analogous to the total pressure $P$ in the stationary channel in that the relative streamlines lie on surfaces of constant $H$ in the rotating case (fig. 2) just as the streamlines lay on surfaces of constant $P$ in the stationary case (fig. 1). From equation (3) it can be seen that

$$\overline{W}\times(\overline{W}\times\bar{\Omega})=\overline{W}\times\nabla H+\frac{2\overline{W}\times(\bar{\omega}\times\overline{W})}{\omega}$$
$$=\overline{W}\times\nabla H+2W^2\frac{\bar{\omega}}{\omega}-2\frac{\bar{\omega}\cdot\overline{W}}{\omega}\overline{W} \tag{5}$$

By means of equation (5), the right side of equation (2) can be expressed as a function of $\overline{W}$, $\bar{\omega}/\omega$, and $\nabla H$.

**Equation of continuity.**—In order to solve for the component of the gradient of $\xi/W$ parallel to the streamline, the equation of continuity is introduced. This equation states

$$\nabla\cdot\overline{W}=0$$

Hence,

$$\nabla\cdot\frac{\xi}{\overline{W}}\overline{W}=\overline{W}\cdot\nabla\frac{\xi}{\overline{W}} \tag{6}$$

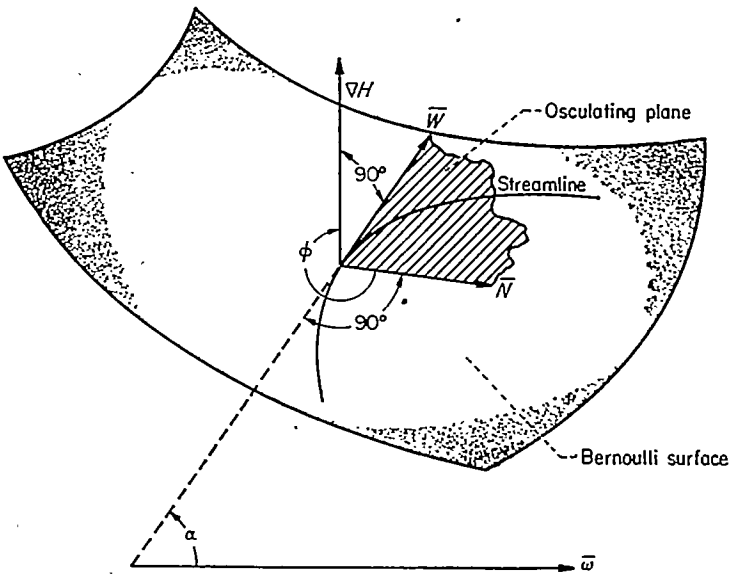



FIGURE 2.—Vector diagram for rotating channel.

and, from equation (5),

$$\nabla\cdot\frac{\overline{W}\times(\overline{W}\times\bar{\Omega})}{\overline{W}\cdot\overline{W}}=\nabla\cdot\frac{\overline{W}\times\nabla H}{W^2}+\overline{W}\cdot\nabla\frac{-2\bar{\omega}\cdot\overline{W}}{\omega W^2} \tag{7}$$

**General differential equation for secondary vorticity.**—Equations (6) and (7) are now substituted into equation (2), and the following differential equation results:

$$\overline{W}\cdot\nabla\frac{\xi}{\overline{W}}=\nabla\cdot\frac{\overline{W}\times\nabla H}{W^2}+\overline{W}\cdot\nabla\frac{-2\bar{\omega}\cdot\overline{W}}{\omega W^2} \tag{8}$$

The first term on the right of equation (8) represents the secondary vorticity growth caused by two effects: (1) the turning of the streamline caused by the rotation of the system, and (2) the turning of the streamline caused by its curvature in the relative system. In order to clarify the physical significance of equation (8), the first term on the right is expanded as shown in appendix B to yield

$$\nabla\cdot\frac{\overline{W}\times\nabla H}{W^2}=\frac{2}{W^2}\nabla H\cdot\frac{\bar{\omega}}{\omega}+\frac{2\kappa_g|\nabla H|}{W} \tag{9}$$

where $\kappa_g$ is the geodesic curvature of the streamline on the Bernoulli surface. Combining equations (8) and (9) results in the following general vector differential equation for secondary vorticity:

$$\overline{W}\cdot\nabla\frac{\xi}{\overline{W}}=\frac{2}{W^2}\nabla H\cdot\frac{\bar{\omega}}{\omega}+\frac{2\kappa_g|\nabla H|}{W}+\overline{W}\cdot\nabla\frac{-2\bar{\omega}\cdot\overline{W}}{\omega W^2} \tag{10}$$

This vector differential equation relates the secondary vorticity $\xi$ at a point along the streamtube to the angular velocity of the system $\bar{\omega}$, the relative velocity of the fluid $\overline{W}$, the geodesic curvature of the streamtube $\kappa_g$, and the gradient of the Bernoulli constant $\nabla H$.

**Physical interpretation of general vector differential equation.**—The first term on the right of equation (10) is influenced by the component of $\nabla H$ parallel to the axis of rotation, and the second term is influenced by the component of $\nabla H$ normal to the principal normal of the streamline. The third term on the right of equation (10) represents the variation of the vorticity component parallel to the streamline caused by the rotation of the system, irrespective of any variation of $H$. This type of secondary motion is discussed in reference 6 for an axial-flow-type channel.

From the first two terms on the right of equation (10), it can be seen that in a rotating radial channel such as in a centrifugal impeller, in which the osculating planes of the relative streamlines are approximately normal to the axis of rotation, the only component of the gradient of $H$ which affects the generation of secondary vorticity is that parallel to the axis of rotation. This corresponds to the component of the gradient of $H$ from the inner to the outer shroud. Thus, the boundary layer on the blade surfaces with the concomitant gradient of $H$ normal to the blade surfaces does not affect the generation of secondary vorticity.

The vorticity component parallel to the streamline induced by the rotation of the system is equal to $-2\omega \cos \alpha$, where $\alpha$ is the angle between the streamline and the direction of the axis of rotation (see fig. 2). The applications in this report are to rotating radial channels so that this term is zero because $\alpha$ is 90°. If the channel is stationary, that is, $\omega$ is equal to zero, equation (10) becomes

$$\overline{W} \cdot \nabla \frac{\xi}{W} = \frac{2\kappa_g |\nabla H|}{W}$$

where it is understood that the variables are in dimensional form because $\omega$ is equal to zero. This is the same equation as that obtained by Hawthorne in reference 2 for a stationary curved channel.

### APPLICATION OF GENERAL VECTOR DIFFERENTIAL EQUATION TO ROTATING RADIAL CHANNELS

In this section the general vector differential equation for secondary vorticity developed in the preceding section is applied to rotating radial channels similar to those in centrifugal-type impellers, an example of which is shown in figure 3, and further simplifying assumptions are made. Numerical examples are solved involving streamtubes with various geometries which follow logarithmic spiral paths, including a straight radial path as a special case. A logarithmic spiral path is similar to the streamlines in a conventional centrifugal impeller and an analytic expression is known for such a path.

**Further assumptions.**—A right-handed orthogonal curvilinear coordinate system with coordinates $u_1$, $u_2$, and $u_3$ is chosen. The unit vectors in the $u_1$-, $u_2$-, and $u_3$-directions are $\bar{i}$, $\bar{j}$, and $\bar{k}$, respectively. The reference point denoted by the subscript $t$ in the general analysis shall refer to the tip of the rotating radial channel so that all linear dimensions are expressed as ratios of the tip radius and all velocities are expressed as ratios of the tip speed $\omega r_t$.

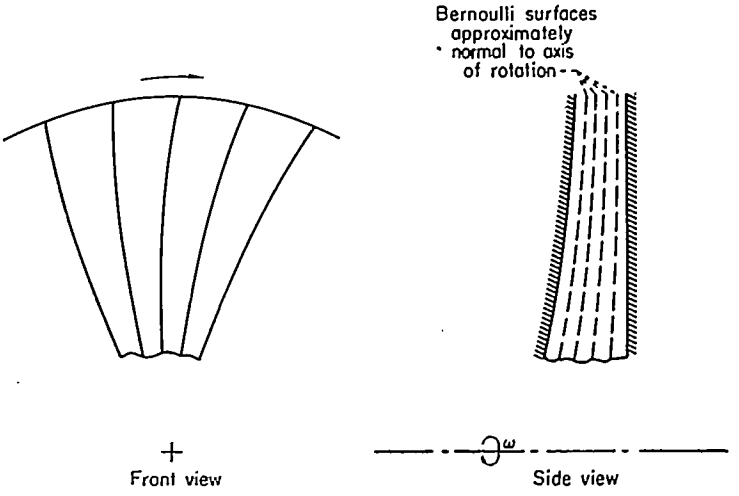



FIGURE 3. Front and side views of centrifugal-type impeller.

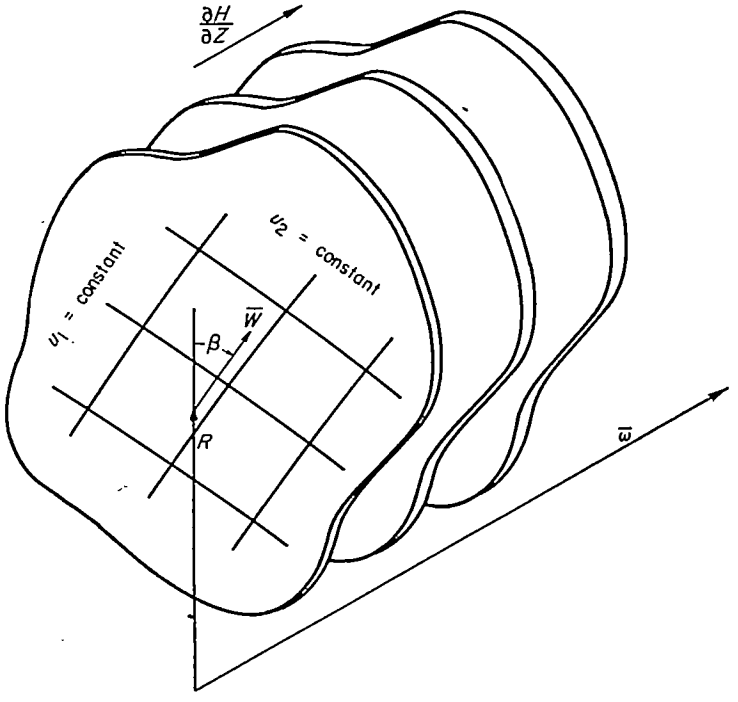



FIGURE 4.—Flow planes for simplified flow in rotating radial channel.

The flow surfaces on which the relative streamlines lie are Bernoulli surfaces. These surfaces are parallel to the $u_1u_2$-surfaces and are assumed to be planes normal to the axis of rotation (fig. 4). As a result of this assumption,

$$\overline{\omega} = \omega \bar{k} \tag{11}$$

and

$$\nabla H = \frac{1}{U_3} \frac{\partial H}{\partial u_3} \bar{k} \tag{12}$$

where the $U_n$ for $n=1, 2, 3$ are the square roots of the components of the metric tensor associated with the orthogonal curvilinear coordinate system employed and are given by

$$U_n=\frac{1}{|\nabla u_n|} \tag{13}$$

in which $\nabla$ is expressed in the coordinates in which the $u_n$ are given. From this assumption it follows that the Bernoulli surfaces are planes coincident with the osculating plane of the relative streamline throughout the channel and $\varphi$ is $+90°$ or $-90°$ when $\left(\frac{1}{U_3}\frac{\partial H}{\partial u_3}\right)$ is positive or negative, respectively. This assumption, which is usual in theoretical studies of secondary flow, implies that the secondary flows are small perturbations on the primary flow. Hence, if the secondary flows are large, the results of this analysis are qualitative rather than quantitative.

The orthogonal curvilinear coordinates in the $u_1u_2$-plane are selected so that lines of constant $u_2$ are streamlines Hence,

$$\overline{W}=W\vec{i} \tag{14}$$

It is not necessary for the application of the method that the coordinate system be so chosen but is merely a matter of convenience. As stated previously, it is necessary to assume a streamline pattern in order to solve for the secondary vorticity. The accuracy of the solution for secondary vorticity is therefore dependent upon the accuracy of the streamline approximation. This approximation might be a two-dimensional potential flow solution, a mean flow path through the channel, potential flow through the stationary channel, and so forth.

Although the Bernoulli surfaces are assumed to be essentially parallel planes, the gradient of the Bernoulli constant is allowed to vary inversely as the assumed streamtube thickness ratio $\tau$ in order to give a closer approximation to conditions in an actual passage. The streamtube thickness ratio $\tau$ is the ratio of the streamtube thickness in the $u_3$-direction at any point to its value at the inlet of the rotating radial channel. Thus,

$$\frac{1}{U_3}\frac{\partial H}{\partial u_3}=\frac{1}{\tau}\left(\frac{1}{U_3}\frac{\partial H}{\partial u_3}\right)_i \tag{15}$$

**Simplified differential equation for secondary vorticity in a rotating radial channel.**—Under the assumptions and definitions given by equations (11), (12), (14), and (15), the differential equation (10) for secondary vorticity in a rotating radial channel becomes

$$\frac{1}{U_1}\frac{\partial}{\partial u_1}\left(\frac{\xi}{W}\right)=\frac{2}{W^3\tau}\left(\frac{1}{U_3}\frac{\partial H}{\partial u_3}\right)_i(1+W\kappa)$$

or along a streamline,

$$d\left(\frac{\xi}{W}\right)=\frac{2}{W^2\tau}\left(\frac{1}{U_3}\frac{\partial H}{\partial u_3}\right)_i(1+W\kappa)\frac{U_1du_1}{W} \tag{16}$$

The term $\xi/W$ is proportional to the secondary circulation, that is, the circulation associated with the secondary vorticity, because $\xi$ is the secondary vorticity and $1/W$ is proportional to the streamtube cross-sectional area normal to the flow and therefore normal to the secondary vorticity vector. It can be seen from equation (16) that the differential change in secondary circulation parameter $d(\xi/W)$ is directly proportional to $\frac{1}{\tau}\left(\frac{1}{U_3}\frac{\partial H}{\partial u_3}\right)_i$, which is the Bernoulli constant gradient. Also, the differential change in the secondary circulation parameter is proportional to the sum $(1+W\kappa)$ and to $U_1du_1/W$. In the sum $(1+W\kappa)$, the term $W\kappa$ is the angular velocity about the axis of rotation of the fluid relative to the rotating system, and the term 1 is the angular velocity of the system itself. Thus, the sum is equal to the absolute angular velocity of the fluid particle. Because the factor $U_1du_1/W$ is the differential element of time required for the fluid particle to pass through the differential channel length $U_1du_1$, the product $(1+W\kappa)\frac{U_1du_1}{W}$ is the differential absolute angle through which the fluid particle is turned. In addition, $d(\xi/W)$ varies inversely with $W^2$ so that large values of $d(\xi/W)$ will occur for small values of $W$.

If the velocity $W$ is constant, the secondary vorticity $\xi$ varies linearly with the secondary circulation parameter $\xi/W$. If the flow is accelerating or decelerating for a given differential change in $\xi/W$, the differential change in $|\xi|$ is greater or less, respectively, than that for constant $W$, as can be seen from the second term on the right in the following expansion of $d(\xi/W)$:

$$d\xi=Wd\left(\frac{\xi}{W}\right)+\frac{\xi}{W}dW \tag{17}$$

However, because the differential change in secondary circulation parameter $d(\xi/W)$ varies inversely with $W$ (see eq. (16)), the tendency of the first term on the right of equation (17) is to produce an effect opposite to that of the second term; that is, accelerating and decelerating flows tend to decrease and increase, respectively, $|\xi|$. Thus, there is a double effect involved so that it is not possible, in general, to predict the exact effect of accelerating or decelerating flow.

**Logarithmic spiral coordinates.**—A logarithmic spiral curvilinear coordinate system is specified in the relative flow planes and a rectilinear coordinate system normal to the flow planes. Hence, the equations of the coordinates are

$$\left.\begin{aligned} u_1&=\ln R+\theta\tan\beta\\ u_2&=\theta-\tan\beta\ln R\\ u_3&=Z\end{aligned}\right\} \tag{18}$$

where $R$, $\theta$, and $Z$ are the usual cylindrical coordinates forming a right-handed system and $\beta$ is the streamtube angle, that is, the angle between the radial and $u_1$-directions, positive when in the direction indicated by positive $\omega$ according to the right-hand rule (see fig. 4). For logarithmic spiral streamtubes, $\beta$ is constant along the streamtube.

The curvature $\kappa$ of a logarithmic spiral streamtube is $\sin\beta/R$, and the differential streamtube length is $\sec\beta\,dR$. Hence, along a logarithmic spiral streamline equation (16) becomes

$$d\left(\frac{\xi}{W}\right)=\frac{2\sec\beta}{W^3\tau}\left(\frac{\partial H}{\partial Z}\right)_i\left(1+\frac{W\sin\beta}{R}\right)dR \tag{19}$$

The following equation results from integrating equation (19) along a relative streamline and noting that $\xi_t$ is equal to zero because the flow was assumed to be initially parallel and uniform except for a total-pressure gradient normal to the osculating plane of the streamline:

$$\xi_x=2W_x \sec\beta\left(\frac{\partial H}{\partial Z}\right)_t\int_{R_t}^{R_x}\frac{1}{W^3\tau}\left(1+\frac{W\sin\beta}{R}\right)dR \quad (20)$$

where $x$ is an arbitrary point along the streamline. This equation can be used to compute the secondary vorticity at any point $x$ along the streamline from the known geometric and flow conditions.

**Case I. Constant $\tau$ and $WR$.**—In Case I,

$$W=\frac{W_t}{R} \quad (21)$$

and the streamtube geometry is similar to the channel in a centrifugal impeller with constant blade height and blade spacing. In this case the central angle subtended by the streamtube width $\Delta\theta$ is constant. Because $\tau$ is constant, $\partial H/\partial Z$ is constant along the streamline. Integrating equation (20) using $W$ as determined by equation (21) yields

$$\xi\Big/\left(\frac{\partial H}{\partial Z}\right)_t=\frac{\sec\beta}{W_tR}\left[\frac{1}{2W_t}(R^4-R_t^4)+\sin\beta(R^2-R_t^2)\right] \quad (22)$$

Equation (22) gives the secondary vorticity $\xi$ at any radius $R$ for Case I. It can be seen that $\xi$ is directly proportional to $(\partial H/\partial Z)_t$ and that the change in $\xi$ between any two radii is dependent on the value of $R_t$.

Several special cases can be considered with this particular geometry. If $R_t$ is equal to zero, equation (22) becomes

$$\xi\Big/\left(\frac{\partial H}{\partial Z}\right)_t=\frac{\sec\beta}{W_t}\left(\frac{R^3}{2W_t}+R\sin\beta\right) \quad (23)$$

If the streamtube walls are straight radial lines, $\beta$ is zero and equation (23) becomes

$$\xi\Big/\left(\frac{\partial H}{\partial Z}\right)_t=\frac{R^3}{2W_t^2}$$

Also, if the channel is stationary, $\bar{\omega}$ is zero and equation (23) becomes

$$\xi\Big/\left(\frac{\partial H}{\partial Z}\right)_t=\frac{\tan\beta}{W}$$

All the quantities in the above equation must be expressed in their dimensional form because $\omega$ is equal to zero.

The condition for $\xi$ to be zero at some radius $R$ is given by

$$\frac{R^2+R_t^2}{2W_t}+\sin\beta=0 \quad (24)$$

as indicated by equation (22).

**Case II. Constant $\tau R$ and $W$.**—In Case II,

$$\tau=\frac{R_t}{R} \quad (25)$$

and the streamtube geometry is similar to the channel in a centrifugal impeller with constant blade spacing ($\Delta\theta$ is constant) and constant flow area (blade height varies inversely with radius). From equations (15) and (25), $\partial H/\partial Z$ is directly proportional to the radius $R$. Integrating equation (20) using equation (25) for $\tau$ yields

$$\xi\Big/\left(\frac{\partial H}{\partial Z}\right)_t=\frac{2\sec\beta}{WR_t}\left[\frac{1}{2W}(R^2-R_t^2)+\sin\beta\,(R-R_t)\right] \quad (26)$$

Equation (26) is the expression for secondary vorticity at any point in the channel for Case II. The change in $\xi$ between any two radii is inversely proportional to the value of $R_t$.

From equation (26) the secondary vorticity is zero at any point in the channel when

$$1+\frac{W\sin\beta}{\left(\frac{R+R_t}{2}\right)}=0 \quad (27)$$

It is interesting to note from equation (27) that the secondary vorticity is zero at a point when the absolute angular velocity (left side of eq. (27)) at the mean radius between the inlet and that point is zero. In the previous discussion of equation (16), it was noted that if $W$ is constant, the differential change in secondary vorticity at a point is zero if the absolute angular velocity of the fluid was zero at that point.

**Case III. Constant $W$ and $\tau$.**—In Case III both the relative velocity $W$ and the streamtube height ratio $\tau$ are constants so that the streamtube geometry is similar to the channel in a centrifugal impeller with constant blade height and constant flow area (blade spacing $\Delta\theta$ varies inversely with radius). Integrating equation (20), having noted that $W$ and $\tau$ are constants, yields

$$\xi\Big/\left(\frac{\partial H}{\partial Z}\right)_t=\frac{2\sec\beta}{W}\left[\frac{1}{W}(R-R_t)+\sin\beta\ln\frac{R}{R_t}\right] \quad (28)$$

Equation (28) expresses the variation in $\xi$ along a streamline for Case III. It can be seen that the change in $\xi$ between any two radii is independent of the value of $R_t$. The condition for zero $\xi$ at a point is

$$1+\frac{W\sin\beta}{(R-R_t)}\ln\frac{R}{R_t}=0 \quad (29)$$

The average absolute angular velocity between the inlet and radius $R$ is given by

$$\frac{1}{R-R_t}\int_{R_t}^{R}\left(1+\frac{W\sin\beta}{R}\right)dR=1+\frac{W\sin\beta}{(R-R_t)}\ln\frac{R}{R_t}$$

so that equation (28) indicates that the secondary vorticity is zero at a point when the average absolute angular velocity (left side of eq. (29)) is zero. This case corresponds closely in geometry to that considered by Hawthorne and others in stationary elbows. The condition for zero $\xi$ in Case III, namely, that the average absolute angular velocity of the fluid is zero, is analogous to the condition of zero turning angle for zero secondary vorticity in a stationary elbow.

**Summary of equations.**—A summary of the geometric and flow conditions as well as the final equations for secondary vorticity and the condition for zero $\xi$ in each case is presented in the following table:

| | Variable | Case I | Case II | Case III |
|---|---|---|---|---|
| Specified conditions | $\Delta\theta$ | Constant | Constant | $\frac{(\Delta\theta)_t}{R}$ |
| | $\tau$ | Constant | $\frac{R_t}{R}$ | Constant |
| | $\frac{\partial H}{\partial Z}$ | Constant | $\frac{R}{R_t}\left(\frac{\partial H}{\partial Z}\right)_t$ | Constant |
| Consequences in flow | $W$ | $\frac{W_t}{R}$ | Constant | Constant |
| | $\xi\Big/\left(\frac{\partial H}{\partial Z}\right)_t$ | Eq. (22) | Eq. (26) | Eq. (28) |
| | Condition for $\xi=0$ | Eq. (24) | Eq. (27) | Eq. (29) |

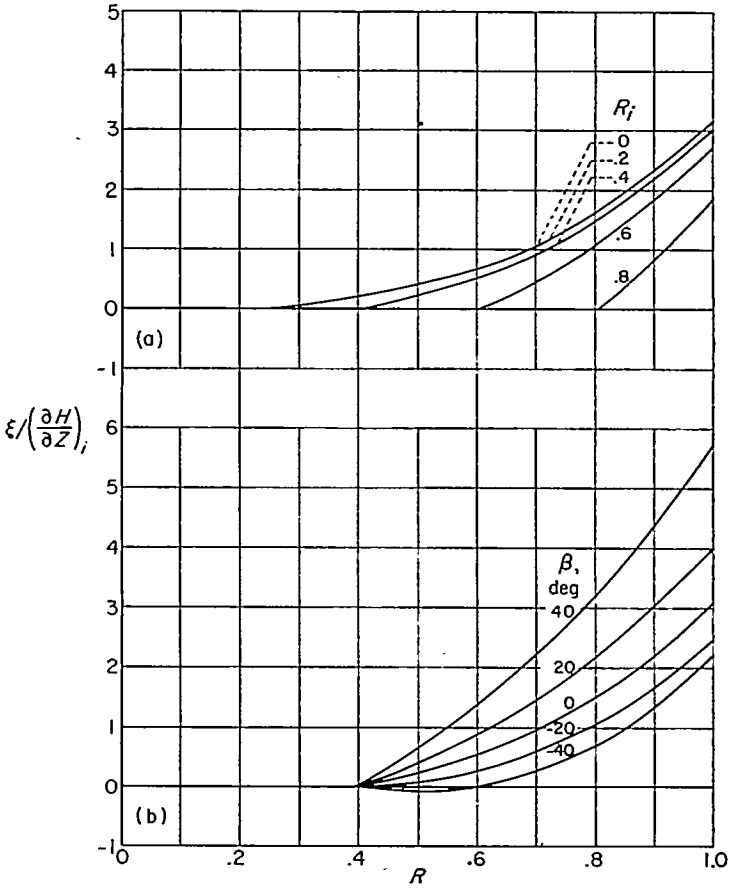
## RESULTS AND DISCUSSION

**Standard conditions.**—The final equations express the secondary vorticity $\xi$ as a function of $(\partial H/\partial Z)_t$, $W_t$, $R$, $R_t$, and $\beta$. As has already been noted, the secondary vorticity is directly proportional to $(\partial H/\partial Z)_t$. The effects of the other variables are presented in a series of figures in which the variation of $\xi/(\partial H/\partial Z)_t$ with two variables is shown with the other variables given their values at standard conditions. These standard conditions are:

| | |
|---|---|
| $W_t$ | 0.4 |
| $\beta$ | 0 |
| $R_t$ | 0.4 |
| $R$ | 1.0 |

**Case I.**—The variation of the secondary vorticity parameter $\xi/(\partial H/\partial Z)_t$ with radius ratio $R$ for various values of the inlet radius $R_t$ and the streamtube angle $\beta$ is shown in figure 5 for Case I. In figure 5 (a) the effect of the inlet radius $R_t$ on the growth of the secondary vorticity in passing through the channel is shown. The variation of the secondary vorticity parameter with inlet radius is small for this case, so the choice of the value of the inlet radius for the standard conditions will not affect the presentation of the results to any considerable extent.

From the curve of $\xi/(\partial H/\partial Z)_t$ against $R$ for various values of streamtube angle $\beta$ in figure 5 (b), it can be noticed that the change in secondary vorticity at the tip radius for a given change in streamtube angle decreases as the streamtube angle decreases. This phenomenon is caused by the fact that although the absolute angular velocity component decreases as $\beta$ decreases from 40° to −40°, the length of path through the channel at first decreases as $\beta$ changes from 40° to 0 but then increases as $\beta$ varies from 0 to −40°. As discussed following equation (16), the change in secondary vorticity varies directly with both the absolute angular velocity and the path length so that when $\beta$ decreases from 0 the effects partially cancel each other. For $\beta$ equal to −40°, the secondary vorticity parameter becomes negative over a range of $R$ following $R_t$ because the absolute angular velocity is initially negative.

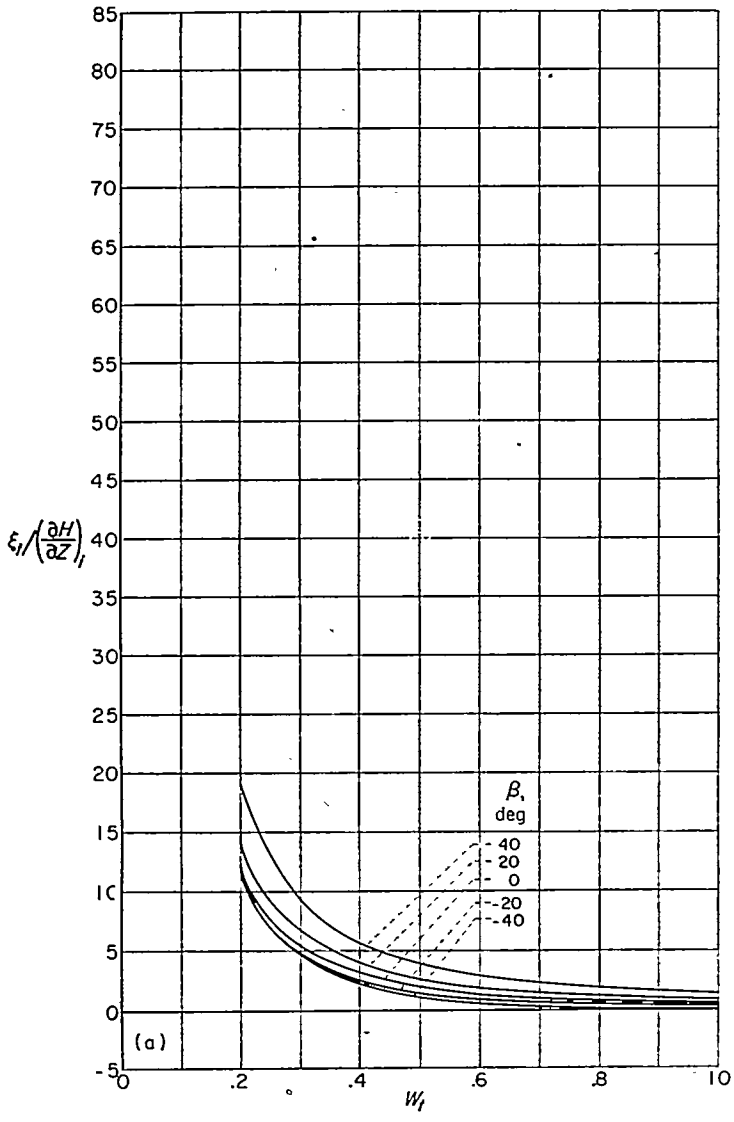



(a) Effect of $R_i$.
(b) Effect of $\beta$.

FIGURE 5.—Variation of $\xi/(\partial H/\partial Z)_t$ with $R$ for Case I. Standard conditions unless otherwise noted.

Figure 6 (a) presents the variation in the secondary vorticity parameter at the tip $\xi_t/(\partial H/\partial Z)_t$ with tip velocity ratio $W_t$ for several values of $\beta$. The effect of $W_t$ is pronounced, as would be expected from equation (16). For $W_t$ equal to 0.2 and $\beta$ equal to −40°, $\xi_t$ is greater than for $W_t$ equal to 0.2 and $\beta$ equal to −20° or 0, because the elongation of the flow path more than balances the effect of reduced absolute angular velocity of the fluid.

**Cases II and III.**—The variation of $\xi_t/(\partial H/\partial Z)_t$ with $W_t$ for several values of $\beta$ is shown in figure 6 (b) for Case II. As in Case I, the values of $\xi_t/(\partial H/\partial Z)_t$ for $W_t$ equal to 0.2 and $\beta$ equal to −40° are greater than the corresponding values for $\beta$ equal to 0 and −20°. The same reason mentioned under Case I applies here. Secondary vorticities are, in general, much greater in this case than in Case I because for the same value of the gradient of the Bernoulli constant $\partial H/\partial Z$ at the inlet, the value of $\partial H/\partial Z$ at any other point in the channel is greater in Case II than in Case I. Again the marked increase in secondary vorticity with decreasing $W_t$ can be noted. Figure 6 (c) presents the same information for Case III as is shown in figures 6 (a) and



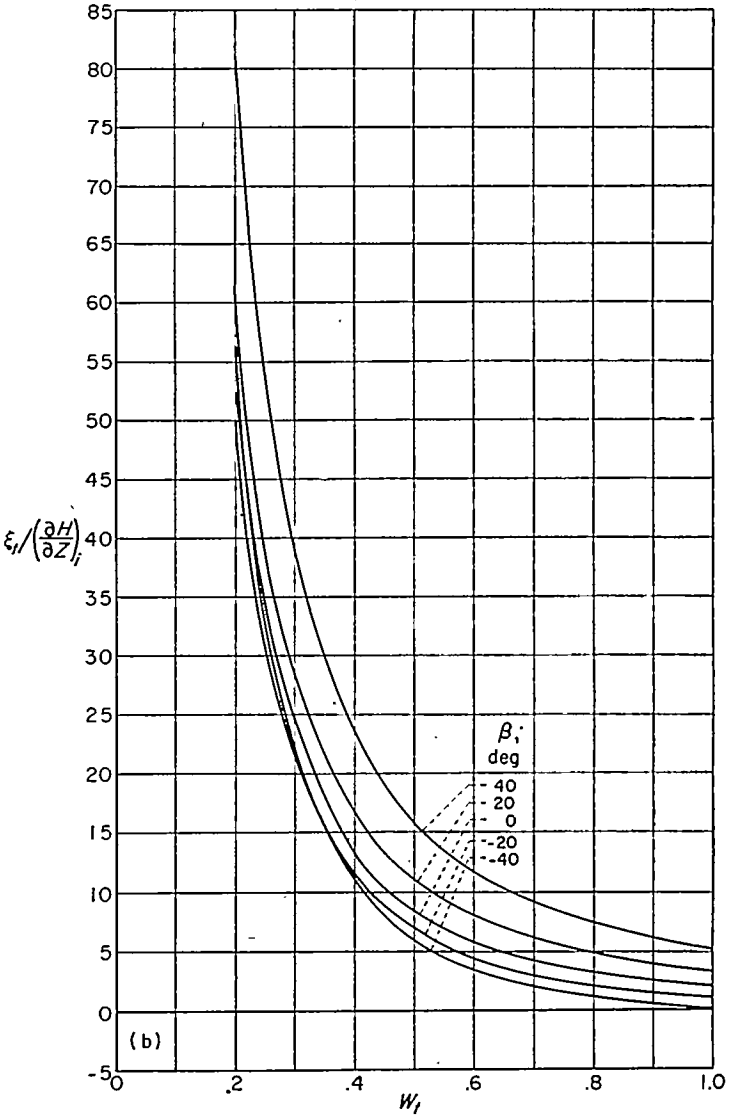

(a) Case I.

FIGURE 6.—Variation of $\xi_i/(\partial H/\partial Z)_i$ with $W_i$. $R_i$ equal to 0.4.

(b) for Cases I and II, respectively. Again the secondary vorticity parameter increases rapidly for small values of $W_i$.

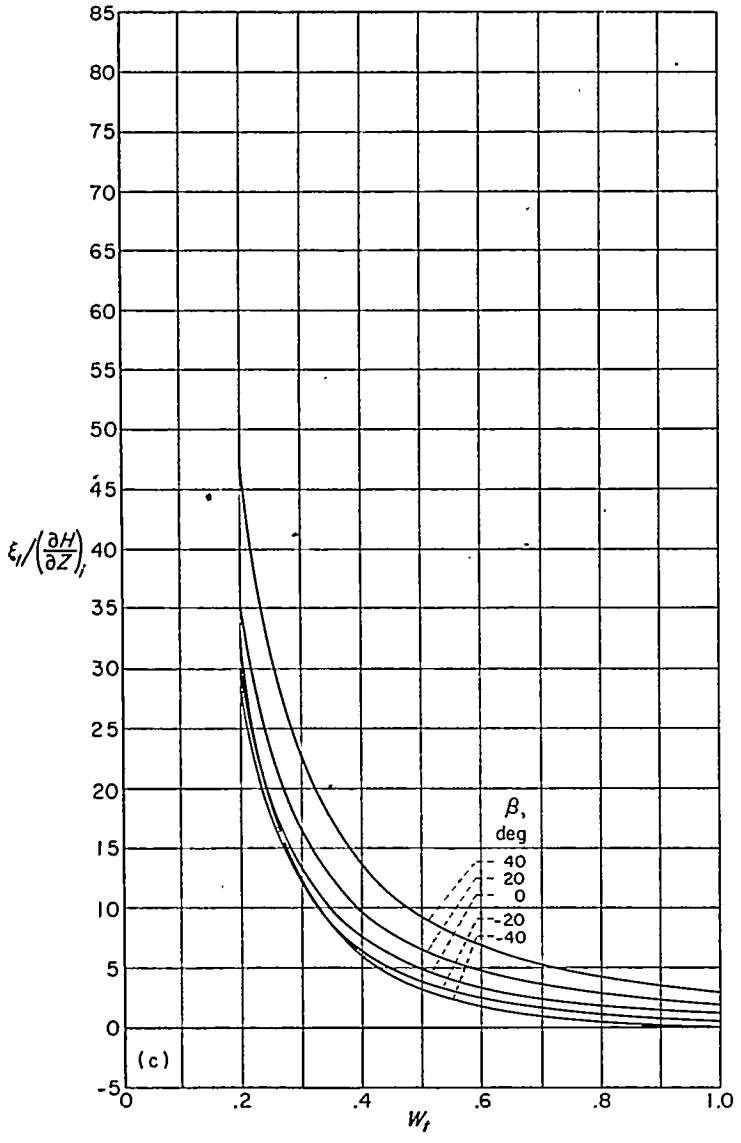
This same trend is evidenced in the experimental results reported by Spannhake in reference 7 (pp. 145–159). The apparatus used for the experiment was an S-shaped tube that was rotated about an axis as shown in figure 7. The fluid, water, enters and leaves in the axial direction along the axis of rotation. A loss coefficient $\lambda$ based on the total-pressure drop from inlet to exit was computed. A range of ratios of circumferential velocity of the pipe to through flow velocity of the water, corresponding to $1/W$, from 0 to 30 was used. The loss coefficient for flow through this apparatus shows the same trend, that is, rapid increase as $1/W$ increases, as that of the secondary vorticity in the theoretical case. Since the experimental work is only somewhat similar to the theoretical work, for in a channel such as the rotating pipe the streamline configuration is much more complex, and since viscous dissipation, not considered in the theoretical work, is an important factor, the only conclusion which can be drawn is that the trends are similar.

**Comparison of Cases I, II, and III.**—A comparison of the variation of $\xi_i/(\partial H/\partial Z)_i$ with $W_i$ is shown by the solid lines in figure 8 for the three cases. It is immediately apparent that the secondary vorticity for Case I is less for all values of $W_i$ than it is for Cases II and III. If the secondary vorticity parameter $\xi_i/(\partial H/\partial Z)_i$ were plotted against inlet velocity $W_i$, the curves for Cases II and III would be the same since the velocity is constant throughout the streamtube. The curve for Case I is shown as the dashed line in figure 8 for which the abscissa is $W_i$. It is seen that the curve for Case I lies above the curves for Cases II and III. As discussed following equation (16), the effect of decelerating flow is to increase the value of $d(\xi/W)$, because of the consequent smaller values of $W$, and at the same time to




(b) Case II.

FIGURE 6.—Continued. Variation of $\xi_i/(\partial H/\partial Z)_i$ with $W_i$. $R_i$ equal to 0.4.




(c) Case III.

FIGURE 6.—Concluded. Variation of $\xi_t/(\partial H/\partial Z)_i$ with $W_t$. $R_i$ equal to 0.4.

decrease $d\xi$ for a given value of $d(\xi/W)$. Thus it appears that of these two effects, that of increasing $d(\xi/W)$, and consequently $d\xi$, predominates because $\xi$ is greater for Case I than for Cases II and III when all three cases have the same inlet velocity, that is, when the velocity is less for Case I than for Cases II and III at all points in the streamtube other than the inlet. When all three cases have the same velocity at the tip, as shown by the solid lines in figure 8, the velocity is greater for Case I than for Cases II and III at all points in the channel other than the tip. Thus $d(\xi/W)$ is smaller and also the flow is decelerating so that, as can be seen from equation (17), these two effects compound, with the result that the secondary vorticity parameter is smaller for Case I than for Cases II and III.

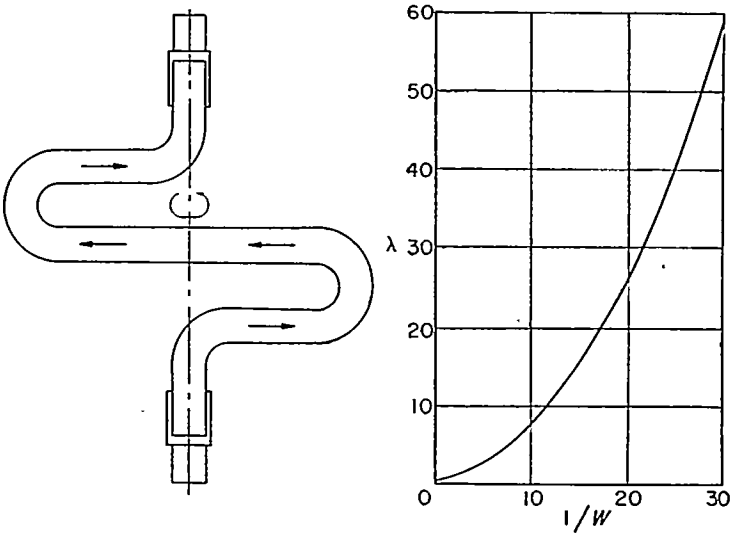



FIGURE 7.—Variation of loss coefficient $\lambda$ with velocity ratio $1/W$ measured in rotating S-shaped tube as reported in reference 7.

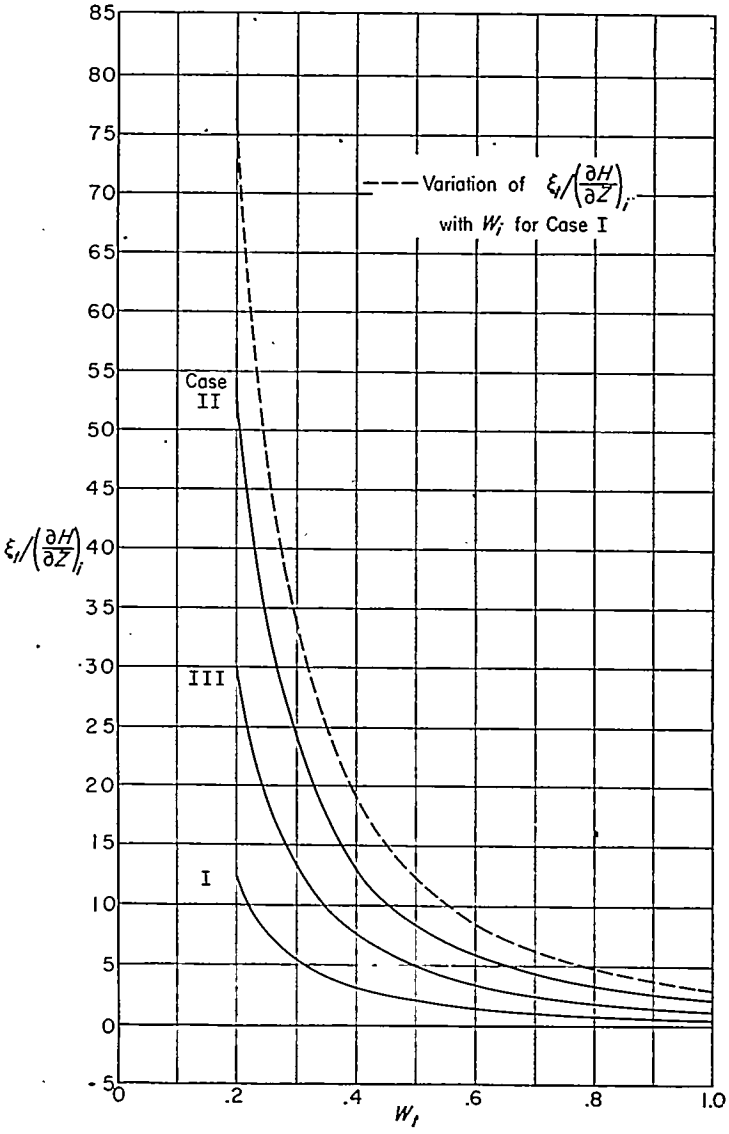



FIGURE 8.—Comparison of variation of $\xi_t/(\partial H/\partial Z)_i$ with $W_t$ for Cases I, II, and III. Standard conditions unless otherwise noted.

The secondary vorticities for Case II are larger than those for Cases I and III because if $(\partial H/\partial Z)_t$ is the same for all three cases, $\partial H/\partial Z$ at any other point is greater for Case II than for Cases I and III. The secondary vorticity, being directly proportional to $\partial H/\partial Z$, is therefore greater for Case II. All three curves show the same pronounced increase in secondary vorticity as $W_t$ decreases.

## SUMMARY OF RESULTS AND CONCLUSIONS

A general vector differential equation for the vorticity component parallel to a streamline in a rotating system is derived. This equation indicates that at a given point along the streamtube the secondary vorticity $\xi$ is a function of the angular velocity of the system $\bar{\omega}$, the relative velocity of the fluid $\overline{W}$, the geodesic curvature of the streamtube $\kappa_g$, and the gradient of the Bernoulli constant $\nabla H$. This equation is then simplified by restricting it to rotating radial channels and by making further simplifying assumptions. This simplified equation was used to solve for the secondary vorticity in three special cases involving different streamtube geometries, and the results are presented in a series of figures. These figures and the equations from which they were obtained indicate that:

1. The differential change in secondary circulation parameter $\xi/W$ at a point in a rotating radial channel is directly proportional to the absolute angular velocity of the fluid, the gradient $\partial H/\partial Z$ of the Bernoulli constant, and the differential element of time required for the fluid particle to pass through the differential channel length. The differential change in $\xi/W$ also varies inversely with the square of the relative velocity.

2. The secondary vorticities are lowest for Case I with constant streamtube thickness ratio $\tau$ and constant central angle subtended by streamtube width $\Delta\theta$ because of the higher fluid velocities upstream of the channel tip.

3. The secondary vorticities are highest for Case II with constant $\tau R$ (where $R$ is radius ratio) and $W$ because the decreasing streamtube thickness downstream of the channel inlet results in increased $\partial H/\partial Z$.

4. The secondary vorticity increased rapidly in all cases as the relative velocity at the tip decreased because of the inverse variation of $d\xi$ with the relative velocity.

5. The decrease in absolute angular velocity of the fluid caused by a decrease in the streamtube angle $\beta$ caused the secondary vorticities to decrease except where the resulting increased path length offset this effect.

6. The loss coefficient measured in an experiment with a rotating S-shaped pipe showed the same trend, that is, rapid increase as the relative velocity ratio $W$ decreased, as evidenced by the secondary vorticity parameter in the theoretical results.

LEWIS FLIGHT PROPULSION LABORATORY
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS
CLEVELAND, OHIO, *August 31, 1953*

# APPENDIX A

## SYMBOLS

The following symbols are used in this report:

| | |
|---|---|
| $H$ | Bernoulli constant relative to rotating system, dimensionless, eq. (4) |
| $\bar{i}$ | unit vector in $u_1$-direction |
| $\bar{j}$ | unit vector in $u_2$-direction |
| $\bar{k}$ | unit vector in $u_3$-direction |
| $\overline{N}$ | principal normal of streamline |
| $P$ | total pressure of fluid in stationary channel |
| $p$ | static pressure, dimensional |
| $R$ | radial length, dimensionless |
| $r$ | radial length, dimensional |
| $U_1, U_2, U_3$ | square roots of components of metric tensor associated with orthogonal curvilinear coordinate system, eq. (13) |
| $u_1, u_2, u_3$ | orthogonal curvilinear coordinates, eq. (18) |
| $V$ | fluid velocity in stationary channel |
| $W$ | velocity relative to rotating system, dimensionless |
| $Z$ | axial distance, dimensionless |
| $\alpha$ | angle between tangent to streamline and direction of axis of rotation |
| $\beta$ | angle between radial line and streamline, positive in direction indicated by positive $\omega$ |
| $\zeta$ | total vorticity in stationary channel |
| $\theta$ | coordinate in right-handed cylindrical coordinate system |
| $\Delta\theta$ | central angle subtended by streamtube width |
| $\kappa$ | curvature of streamline |
| $\kappa_g$ | geodesic curvature of streamline |
| $\lambda$ | loss coefficient |
| $\xi$ | vorticity component parallel to streamline, dimensionless |
| $\rho$ | fluid mass density, dimensional |
| $\tau$ | ratio of streamtube height in axial direction at any point to that at inlet |
| $\varphi$ | angle between principal normal of streamline and normal to Bernoulli surface, positive when indicating positive rotation about streamline according to right-hand rule |
| $\psi'$ | angle between vortex line and streamline in stationary channel |
| $\Omega$ | total relative vorticity, dimensionless |
| $\omega$ | angular velocity of system, dimensional |
| $\nabla$ | vector operator |

Subscripts:

| | |
|---|---|
| $n$ | index equal to 1, 2, 3, denoting components in $u_1$-, $u_2$-, $u_3$-directions, respectively |
| $i$ | inlet |
| $t$ | characteristic point in channel which is tip in case of rotating radial channel |
| $x$ | arbitrary point along streamtube |

Superscript:

| | |
|---|---|
| — | vector quantity |

## APPENDIX B

### EXPANSION OF $\nabla \cdot \frac{\overline{W} \times \nabla H}{W^2}$

In order to simplify equation (8) and bring out its physical significance, the term $\nabla \cdot \frac{\overline{W} \times \nabla H}{W^2}$ is expanded as follows:

$$\nabla \cdot \frac{\overline{W} \times \nabla H}{W^2} = \frac{-\nabla W^2}{W^4} \cdot (\overline{W} \times \nabla H) + \frac{1}{W^2} \nabla H \cdot \overline{\Omega} \qquad \text{(B1)}$$

This expression is further simplified by expressing the first term on the right as

$$-\frac{\nabla W^2}{W^4} \cdot (\overline{W} \times \nabla H) = \frac{-2}{W^4} [(\overline{W} \cdot \nabla \overline{W}) \cdot (\overline{W} \times \nabla H) + (\overline{W} \times \overline{\Omega}) \cdot (\overline{W} \times \nabla H)]$$

Since the streamlines lie on surfaces of constant $H$ (see fig. 2), $\nabla H$ is normal to $\overline{W}$ and the preceding equation becomes

$$\frac{-\nabla W^2}{W^4} \cdot (\overline{W} \times \nabla H) = \frac{-2}{W^4} [\overline{W} \cdot \nabla \overline{W}) \cdot (\overline{W} \times \nabla H) + W^2 \nabla H \cdot \overline{\Omega}] \qquad \text{(B2)}$$

Equations (1) and (3) and the fact that $\nabla H$ is normal to $\overline{W}$ are used to simplify the term $\nabla H \cdot \overline{\Omega}$ as follows:

$$\nabla H \cdot \overline{\Omega} = \nabla H \cdot \left[ \frac{\xi}{\overline{W}} \overline{W} + \frac{2 \left( \frac{\overline{\omega} \times \overline{W}}{\omega} \right) \times \overline{W} + \nabla H \times \overline{W}}{\overline{W} \cdot \overline{W}} \right] = -2 \nabla H \cdot \frac{\overline{\omega}}{\omega} \qquad \text{(B3)}$$

Combining equations (B1), (B2), and (B3) yields

$$\nabla \cdot \frac{\overline{W} \times \nabla H}{W^2} = \frac{-2}{W^4} (\overline{W} \cdot \nabla \overline{W}) \cdot (\overline{W} \times \nabla H) + \frac{2}{W^2} \nabla H \cdot \frac{\overline{\omega}}{\omega} \qquad \text{(B4)}$$

The first term on the right of equation (B4) involves a triple scalar product in which the term $(\overline{W} \cdot \nabla \overline{W})$ represents the relative acceleration of the fluid particle. This acceleration has two components, one tangent to the streamline and the other along the principal normal equal to $W^2 \kappa$, where $\kappa$ is the curvature of the streamline. Hence, the triple product is equal to $-W^3 |\nabla H| \kappa \sin \varphi$, where $\varphi$ is the angle measured from the principal normal of the streamline to the normal to the Bernoulli surface (in the direction of the gradient of $H$), positive when indicating a positive rotation about the streamline according to the right-hand rule. The quantity $\kappa \sin \varphi$ is the geodesic curvature $\kappa_g$ of the streamline, so that this term is zero when the streamlines are geodesics on the Bernoulli surface. Hence, equation (B1) becomes

$$\nabla \cdot \frac{\overline{W} \times \nabla H}{W^2} = \frac{2}{W^2} \nabla H \cdot \frac{\overline{\omega}}{\omega} + \frac{2 \kappa_g |\nabla H|}{W} \qquad (9)$$

Equation (9) is used in the simplification of equation (8) to clarify the physical significance of the various terms of equation (8).

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